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UNIVERSIDADE TÉCNICA DE LISBOA
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MESTRADO
MATHEMATICAL FINANCE

TRABALHO FINAL DE MESTRADO
DISSERTAÇÃO

CLASSIFICATION OF 2X2 FICTITIOUS PLAY

MARIA FERREIRA VILA LUZ

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ORIENTAÇÃO:
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Resumo

Teoria de jogos é uma teoria no ramo da matemática criada com o objectivo de estudar e modelar eventos onde duas ou mais pessoas interagem entre si. O objectivo desta teoria é perceber como fazer escolhas óptimas quando se está perante um conflito. Existem vários tipos de jogos, no entanto, ao longo deste trabalho final de mestrado, focámos o nosso estudo num tipo particular de jogos, os jogos evolucionários.

Este é um tipo de jogo, onde ao longo do tempo, as estratégias de cada jogador se adaptam e convergem, em geral, para um equilíbrio. Desta forma, os jogadores não precisam de agir racionalmente. Um caso especial dos jogos evolucionários são os jogos fictícios entre dois jogadores, cada um com duas estratégias.

O nosso objetivo foi precisamente demonstrar que ao longo do tempo existe a convergência do processo adaptativo para um equilíbrio. Para melhor compreender este processo de convergência, criámos uma família de exemplos onde sintetizámos todos os jogos fictícios entre duas pessoas com duas estratégias cada.

Palavras-chave: Teoria de jogos, Jogos Fictícios, Dinâmicas da melhor resposta, Equilíbrio de Nash, Classificação

Abstract

Game Theory is a mathematical theory that emerged with the aim of studying and modelling events between two people. The goal of this theory is to understand how to make the best choice of strategy when we are facing a conflict. There are many types of games but throughout this dissertation we focused in evolutionary games. These are games that are repeatedly infinitely. Over time there is a dynamic adaptation of the strategies, the equilibrium comes naturally and therefore, the players do not use their rationality. Within this type of games there are a particular game between two people which is known as fictitious play. We focused our study in these type of games when both of the players have two strategies. Our goal was to show that, over time, the system will converge to the equilibrium. In order to better understand this interaction we developed a family of examples where we synthesized all the possible combinatorially types.

Keywords: Game theory, Fictitious play, Best response dynamics, Nash equilibrium, Classification

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Chapter 1

Introduction

At any moment the economic agents, families and companies have to make decisions. In several interaction situations, the agents are forced to act strategically.

Game theory leads with strategic interactions among multiple (2 or more) decision makers, to which we usually call players. For every game, each one of the players has an objective function associated, which is defined according to their own preferences. We assume that the players are rational and so, they are able to order their strategies given their conjectures or beliefs on how the other players are going to play. If they get more utility maximizing the objective function, this one is called utility function or a benefit function. On the other hand, if their utility is greater when they minimize the objective function, we can call that cost function or lost function. Usually, the objective function of a player depends on the choices of the other players. Consequently, a player cannot simply optimize his own objective function independently of the choices of the other players.

In other words this means that the decisions made by one of the agents are conditioned by the behaviour or expected behaviour of the opponent. Game theory consists exactly in the study of how the agents react when they are confronted into these interaction situations. In order to maximize his payoff, each player needs to examine carefully the game conditions. It is required to identify his and the opponent available strategies, to analyse which are the results, associated to each strategy combination, to check the available information to each participant and to know the specific moments when the decisions are made. The game theory can be applied into several fields from economics to biology through social relations. Summing up, any situation where the agents get into a direct relation in order to achieve certain results.

To define a game we need the number of agents, the number of strategies that each one of the agents has and the payoff associated to each strategy combination.

To a more comprehensive introduction to the fundamentals of game theory see one of the earliest work that started game theory in its modern form, Morgenstern & von Neumann. (1944).

There are three types of games: static, sequential and evolutionary games.

Static games are when the participants make their choices simultaneously (see Brown (1951)).

Sequential games, which are dynamic games since strategies are not chosen simultaneously but rather sequentially. Here the dynamics is limited to the reaction that a player has at a given moment against the action of another or other agents in the previous time period. In this case there is no dynamic rule able to determine the whole course of the game.

Evolutionary games are effectively dynamic games, (see Hofbauer & Sigmund (1998)). They rely on a mechanism that allows us to understand how the strategies can evolve over time. In these games there is one more important element to have into consideration, a dynamic rule that can change payoff and the behaviour of the players over time. Here we are expecting that, after a transitional period, the game will converge to a dominant long-term equilibrium. In that point, the agents must have adopted an evolutionary stable strategy, meaning, a strategy that they will no longer abandon unless some external force disrupts the game's underlying conditions.

The major difference between static game and evolutionary game is that the first one admits players as completely rational, able to identify immediately the dominant strategy. Contrariwise, the second one predicts a gradual adjustment, in which the set of players will progressively change to the dominant strategy, so that the equilibrium is only reached after the dynamic transition phase has been exhausted.

Throughout this dissertation our main focus will be in this last type of games. If the game theory is defined as the science that studies the strategic behaviour, the theory of evolutionary games is the science that studies the robustness of strategic behaviour. In evolutionary games there is an implicit acknowledgement that the agents learn and evolve over time. The strategy that they choose from beginning of the game can be different the strategy that maximizes their payoff. The systematic interaction with the opponents will take them to modify their behaviour over time. We focused our study in a specific case of evolutionary games, called Fictitious

Play. It was introduced by George W. Brown in 1951 (see Brown (1951)). It consists in two players having a finite number of pure strategies. They play repeatedly in such a way that at each round, they use a myopic pure best response against the empirical strategy distribution of his opponent. Fictitious Play can be in discrete-time or in continuous-time.

Initially, in Chapter 2, we will present some basic concepts and notations about game theory. These will be very useful to better understand the rest of our work. Our focus will be on two player games given by a bimatrix.

Next, in Chapter 3, we introduce the learning dynamics called Fictitious Play and we go deeper into this subject. Here we present and discuss a geometric representation of the strategy space.

Chapter 4 is the main section of this entire work. Here we analyse the case of 2×2 fictitious play. We guide our study using two specific games, "Prisoner's dilemma" and "Matched Pennies" both of them between two players each one with two strategies. Based on these examples we will produce an overview of classical results on fictitious play dynamics. In the end of this chapter we will be able to characterize all combinatorial types of 2×2 fictitious play.

Chapter 2

Basics concepts in Game Theory

In this chapter we will present some basic notations from game theory and fictitious play. (see Ostrovski (2013))

Definition 1. A finite game in normal form is a tuple $\Gamma = \left(\mathcal{I}, \left\{S^i\right\}_{i\in\mathcal{I}}, \left\{u^i\right\}_{i\in\mathcal{I}}\right)$, where:

1. $\mathcal{I} = \{1, \cdots, N\}, N \in \mathbb{N}$, is a (finite) collection of players;
2. $S^i = \{1, ..., n_i\}, n_i \in \mathbb{N}$ is the (finite) collection of *pure strategies* of player $i \in \mathcal{I}$;
3. $u^i : S^1 \times ... \times S^N \to \mathbb{R}$ is the *payoff function* of player $i \in \mathcal{I}$.

$S = S^1 \times \cdots \times S^N$ is the space of strategy tuples and $s \in S$ is an element of S that represents a *(pure) strategy profile*.

Interpreting the previous definition, in the game Γ, each player $i \in I$ chooses one of his available strategies $s_i \in S^i$, independently and without knowing beforehand the choices of the opponents. Then, each player i gets a payoff $u^i(s_1, ..., s_N)$. The payoff depends on his and his rivals' strategies. The goal of the competitors is to maximize their own outcome.

We can enlarge the discrete set of pure strategies to a set of mixed strategies. Denote by Σ^i the set of mixed strategies of player i. For $i \in \mathcal{I}$, we consider

$$\Sigma^i := \left\{ x \in \mathbb{R}^{n_i} : x_k \geq 0, \sum_{k=1}^{n_i} x_k = 1 \right\}$$

for the probability distributions over a player's pure strategies. Notice that, geometrically, Σ^i is a $(n_i - 1)$ dimensional simplex in $\mathbb{R}^{n_i}$. We consider $\Sigma = \Sigma^1 \times ... \times \Sigma^N$ and we assume $\sigma \in \Sigma$ as a (mixed) strategy profile.

Given this we are able to extend the payoff functions to consider mixed strategies. Denote this functions by $\tilde{u}^i : \Sigma \to \mathbb{R}$. Let $\sigma = (\sigma_1, ..., \sigma_N) \in \Sigma$ and consider that each $\sigma_i \in \Sigma^i$ is a probability distribution over the pure strategies of player i, this is, $\sigma_i = \left(\sigma_i^1, ..., \sigma_i^N\right)$. So we can write

$$\tilde{u}^i(\sigma) = \tilde{u}^i(\sigma_1, ..., \sigma_N) := \sum_{k_1=1}^{n_1} ... \sum_{k_N=1}^{n_N} \sigma_1^{k_1} ... \sigma_N^{k_N} u^i(k_1, ..., k_N)$$

This can be seen as the expected payoff to player i, if each one of the opponents is randomising over his strategies giving to his mixed strategy σ_i. Notice that, in order to simplify the presentation, henceforth we will use always u^i either pure strategies or mixed strategies.

2.1 Two Player Games

In this dissertation our main focus will be on the two-player games that can play mixed strategies (see Shapley (1964)). We will consider two players, player A which has n_1 pure strategies and player B which has n_2 pure strategies. Just for simplify we will consider that $n_1 = m$ and $n_2 = n$.

We define $e_i \in \Sigma^A, i \in \{1, ..., m\}$ as a standard unit vector correspondent to the first player's strategy and $e_j \in \Sigma^B, j \in \{1, ..., n\}$ to the second player's strategy. Let A and B denote the payoff matrices of player A and player B, respectively. Taking this into account, the (i, j) entry of matrix A can be given by $a_{ij} = e_i^\top A e_j = u^A(i, j)$ and, in the same way, the (i, j) entry of matrix B is given by $b_{ij} = e_i^\top B e_j = u^B(i, j)$. Theoretically, a_{ij}, represents the payoff for player A when he chooses to play the strategy i against strategy j chosen by player B. Along the same line, b_{ij}, gives us the payoff for player B when he picks strategy j and A picks strategy i. Otherwise stated, a_{ij} and b_{ij} are the respective payoffs of the pure strategy profile (i, j) to the players A and B respectively.

We assume $u^A : \Sigma^A \to \mathbb{R}$ and $u^B : \Sigma^B \to \mathbb{R}$ as a linear functions. By linearity, this means that the expected payoff of the mixed strategy profile $(p, q) \in \Sigma^A \times \Sigma^B = \Sigma$ are $u^A(p, q) = p^\top A q$ for player A and $u^B(p, q) = p^\top B q$ for player B. This payoff functions can be represented by a *bimatrix*, which is, a tuple of matrices (A, B) where $A, B \in \mathbb{R}^{m \times n}$. A finite two-player game given in this form (A, B) is called a *bimatrix game*.

Along this dissertation, we will assume vectors $p \in \Sigma^A$ as row vectors in $\mathbb{R}^{1\times m}$, so that for $(p,q) \in \Sigma$ we can write:

$$u^A(p,q) = pAq \text{ and } u^B(p,q) = pBq$$

Up to here we have introduced the dynamics of two player games. Next let us introduce some strategic notions taking into account that players are rational and how that could affect the form that they see the game. We will also evaluate their set of strategies and how can they maximize their own payoff.

2.1.1 Nash Equilibrium

Definition 2. The *best response correspondences* are the ones that assign payoff-maximising response strategies to any given strategy of a player's opponent, i.e., $\mathcal{BR}_A : \Sigma^B \to \Sigma^A$ and $\mathcal{BR}_B : \Sigma^A \to \Sigma^B$ defined by

$$BR_A(q) := \arg\max_{\bar{p}\in\Sigma^A} (\bar{p}Aq) \quad \text{and} \quad BR_B(p) := \arg\max_{\bar{q}\in\Sigma^B} (pB\bar{q})$$

The **maximal-payoff** functions are

$$\begin{cases} \bar{A}(q) & := \max\limits_{\bar{p}\in\Sigma^A} (\bar{p}Aq) \\ \bar{B}(p) & := \max\limits_{\bar{q}\in\Sigma^B} (pB\bar{q}) \end{cases}$$

$$\text{So that,} \quad \begin{cases} \bar{A}(q) = u^A(\bar{p},q) \text{ for } \bar{p} \in BR_A(q) \\ \bar{B}(p) = u^B(p,\bar{q}) \text{ for } \bar{q} \in BR_B(p) \end{cases}$$

Notice that the maximal payoff of A knowing B's strategy, q, correspond to the maximal entry of the vector Aq. The same can be checked for the maximal payoff to player B.

Generically, the best response correspondences, $BR_A : \Sigma^B \to \Sigma^A$ and $BR_B : \Sigma^A \to \Sigma^B$ are almost everywhere single-value. There are some exceptions for a finite number of hyperplanes that we will present you soon.

In the cases, where the value taken by BR_A is single, it is represented by a standard unit vectors $e_i, i = 1, 2, ..., m$, which corresponds to a pure strategy of player A. On the other hand, when BR_A is not a unique value, it is a set of convex combinations of subset of $e_i, i = 1, 2, ..., m$. The analogous holds for player B.

The sets Σ^A and Σ^B can be partitioned, respectively, into n and m parts. Consider R_i^A as the *preference region of strategy i for player A* and, similarly,

R_j^B as the *preference region of strategy j for player B*. In other words we can say that the region R_j^B is a set of strategies of player A (belongs to Σ^A) to which the best response of player B is strategy j. This means, the strategy j is the one where B expects to get the highest payoff when A chooses to play a strategy in R_j^B.

$$\begin{cases} R_i^A := BR_A^{-1}\left(e_i\right) \subseteq \Sigma^B \text{ for } i = 1, ..., m \\ R_j^B := BR_B^{-1}\left(e_j\right) \subseteq \Sigma^A \text{ for } j = 1, ..., n \end{cases}$$

We use the notation $R_{ij} := R_j^B \times R_i^A$ for $j = 1, ..., n$ and $i = 1, ..., m$ for the subsets of Σ when A and B have a fixed strategy preference.

Definition 3. For a generic game (A, B) a codimension-one hyperplane of Σ^A is a subset of Σ^A where BR_B contains two different pure strategies e_j and $e_{j'}$ and all respective convex combinations. To these hyperplanes we call indifference sets and we represent them as $Z_{jj'}^B$. The same happens for $Z_{ii'}^A$. The formal definition is

$$Z_{ii'}^A := \left\{q \in \Sigma^B : (Aq)_i = (Aq)_{i'} \geq (Aq)_k \, \forall k = 1, ..., m\right\}$$
$$Z_{jj'}^B := \left\{p \in \Sigma^A : (pB)_j = (pB)_{j'} \geq (pB)_l \, \forall l = 1, ..., n\right\}$$

or equivalently,

$$Z_{ii'}^A = R_i^A \cap R_{i'}^A \subseteq \Sigma^B \quad \text{and} \quad Z_{jj'}^B = R_j^B \cap R_{j'}^B \subseteq \Sigma^A$$

for $i \neq i'$ and $j \neq j'$.

Interpreting this mathematical terminology, $Z_{jj'}^B$ is a subset of Σ^A forming the boundary of two distinct regions R_j^B and $R_{j'}^B$. For player B, when A chooses to play a strategy p placed in this border subset, is indifferent to choose strategy j or j'. Both of them give him the same utility and that utility is the maximum utility he achieves in response to strategy p of A.

Definition 4. A (mixed) strategy profile $(\bar{p}, \bar{q}) \in \Sigma$ is called a *Nash equilibrium*, if

$$\bar{p} \in BR_A\left(\bar{q}\right) \text{ and } \bar{q} \in BR_B\left(\bar{p}\right)$$

If a Nash equilibrium lies inside Σ, it is called *completely mixed*, otherwise it is called pure.

This notion was firstly introduced by John Nash in 1950. To better understand the behaviour of the players. (see Jr. (1951))

The main idea of this concept is that, in Nash equilibrium the optimal outcome of a game is one where no player has a motivation to deviate from his chosen strategy after considering a rival's choice. In general, an individual can receive no incremental benefit from changing actions, assuming other players remain constant in their strategies.

Suppose that player A pick the strategy i and player B choose strategy j. We say that the pair (i, j) is a Nash Equilibrium if and only if i is the best response to j and simultaneously j is the best response to i. When this happens, neither player has motivation to switch to a different strategy. Note that this doesn't mean that both players are getting the highest possible payoff. This means that they are both receiving the highest possible payoff given their opponent's strategy.
The Nash equilibrium may be not unique. The set of Nash equilibrium may consists of discrete or continuous points in Σ.

It was proved also by Nash (see Jr. (1951)) that the set of Nash equilibrium is non-empty for every bimatrix game. The proof of the existence of a Nash equilibrium is a classical application of Kakutani fixed point Theorem for correspondences (see Ok (2007))

The following lemma gives a simple characterization of a completely mixed Nash equilibrium. Its proof can be found in Osborne (2003).

Lemma 1. *The point $\left(E^A, E^B\right) \in int\,(\Sigma)$ is a (completely mixed) Nash equilibrium of an $m \times n$ bimatrix game (A, B) if and only if, for all $i, i' = 1, ..., m$ and $j, j' = 1, ..., n$,*

$$\left(AE^B\right)_i = \left(AE^B\right)_{i'} \text{ and } \left(E^A B\right)_j = \left(E^A B\right)_{j'}$$

This Lemma says that when B plays the strategy E^B, player A gets the same payoff whatever strategy he uses. The same happens to player B when A plays E^A. Summing up, in response to E^B, player A is indifferent between all his strategies and similarly player B is also indifferent between all his strategies when A chooses play E^A. This Lemma also ensures that $E^A \in R_j^B$ and $E^B \in R_i^A$ for all i, j. So, $(E^A, E^B) \in R_{ij}$ and $E^A \in Z_{jj'}^B$, $E^B \in Z_{ii'}^A$ for all i, i', j, j'. Geometrically, this Lemma represents the intersection of all indifference regions.

Example 1 (Prisoners Dilemma)**.** Two prisoners (A and B), suspected of theft, are taken into custody. However, cops don't have enough evidence to sentence them of that crime but only to sentence them on the charge of possession of stolen goods.

The cop will examine their answers on separate rooms, this means that the prisoners can't talk to each other (so we are in the presence of imperfect information). Each one of them has two possible answers, they can lie or confess. In the first one they are cooperating to each other saying that they don't stole the goods, in the second one they assume the crime. We can find four possible cases. First, if they cooperate with each other, this means that none of them confesses the crime (Lie,Lie), they will both be charged the lesser sentence, a year of prison each. Second possible situation is when both prisoners confess the crime, in this case each prisoner will be sentenced to two years. The third and fourth situation is when one of them confesses the crime and the other one does not. The cop wants that each of the prisoners confess the crime, for this he offers them an incentive, a 'get out of jail free card', while the other prisoner that doesn't confess will be sentenced to a three years term.

Both of them receives the same deal and they know the consequences of each action. They know also that the other prisoner receives the same conditions, we say that we have complete information. Since two players cannot contact with each other and assuming that they will make the decision at the same time, this game can be considered as a simultaneous game, and can be analysed using the strategic form, looking to the correspondent bimatrix, which synthesized the four possible situations.

Prisoner B

$$\text{Prisoner A} \qquad \begin{array}{cc} & \begin{array}{cc} \quad Lie \quad & \quad Conf \quad \end{array} \\ \begin{array}{c} Lie \\[1em] Conf \end{array} & \begin{pmatrix} \begin{bmatrix} & 1y \\ 1y & \end{bmatrix} & \begin{bmatrix} & 0y \\ 3y & \end{bmatrix} \\ \begin{bmatrix} & 3y \\ 0y & \end{bmatrix} & \begin{bmatrix} & 2y \\ 2y & \end{bmatrix} \end{pmatrix} \end{array}$$

Interpreting this bimatrix we can check the four cases described before. So, if neither of them confesses (Lie,Lie), they will be penalized one year each. If both players confess the crime (Confess,Confess), they will be penalized a two years sentence each. In the other hand, if only one confesses (Lie,Confess) or (Confess,Lie), the prisoner who confesses will go free, while the other will be penalized a three years sentence. These can be seen as the respective payoff for each set of strategies.

As we said before, in order to solve this problem each prisoner will evaluate their best strategy according to their own benefit taking into account the other prisoner's possible strategies. Obviously, they have a greater benefit the less years they are in jail. By eliminating all dominated strategies, we can get the dominant strategy.

Before thinking about what each player is going to do in this situation, let us present the individual payoff matrix for each one of the players.

We get for player A and B, respectively,

$$A = \begin{bmatrix} -1 & -3 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ -3 & -2 \end{bmatrix}$$

For choosing his best strategy, player A has to have in consideration the choice of player B. So he has to build a belief about what strategy player B is going to make. Looking at the payoff matrix of player A we see that if player B lies (first column, $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$), A stays better, this means, maximizes your own payoff, if he plays "confess". Since if he lies he would be arrested one year instead of remaining free. In the other hand, if B confesses (corresponds to second column, $\begin{bmatrix} -3 \\ -2 \end{bmatrix}$), prisoner A will choose confess, because with that choice he gets stuck two years instead of three, what would happen if he lies. Given this, we can conclude that the best response for player A is always confess, regardless of which strategy B chooses.

We can apply the same logic to player B, looking at his payoff matrix. If A lies, the best response of B will be confess and if A confesses the best response of B will be confess as well. So, the rational thing to do for player B is also to confess.

Therefore, 'to confess' is the dominant strategy. Since, (Confess, Confess) is the set of strategies that maximises each prisoner's utility given the other prisoner's strategy. It is the Nash Equilibrium of this game.

But here is the dilemma. As we seen before, Nash Equilibrium can be used to anticipate the result of the finite games, whenever such equilibrium exists. Both prisoners are choosing to protect themselves at the expense of the other participant, but as a result of this purely logical thought process, in the end we will face with a situation where the two players find themselves in a worse state than if they had cooperated with each other in the decision-making process. Here the Nash Equilibrium does not meet the criteria for being Pareto optimal. In other words, the individual choice of each one of the players is to lie. As we have seen, when this happens both of them get stuck for two years. However, this is not the best joint option because if the two players confess, they only get stuck one year each instead of two.

The prisoner's dilemma is an abstraction of common situations where the choice of the best individual leads to both of them confess down, whereas if they both lying they would provide better results. It is said that this dilemma has an 'inefficient balance' because the scheme of incentives and rationality leads to worse results.

Sometimes it is not so simple to find out the Nash equilibrium, (p^*, q^*). Therefore, we will show how to get it analytically.

Considering that p^* is the best response of player A, whatever B's move, and q^* is the best response of B, whatever A plays, thus we have,

$$p^* \in \mathcal{BR}_A\left(q^*\right) \quad \text{and} \quad q^* \in \mathcal{BR}_B\left(p^*\right)$$

To find the Nash Equilibrium, first we have to compute $\mathcal{BR}_A$ and $\mathcal{BR}_B$. Let us deduce the explicit expression for both of them.

$$\begin{aligned}\mathcal{BR}_A\left(q\right) &= \arg\max_p \left(pAq\right) = \arg\max_p \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} -1 & -3 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \\ &= \arg\max_p \left(-p_1 q_1 - \left(3p_1 + 2p_2\right) q_2\right) = \\ &= \arg\max_p \left(-p_1 - 2 + 2q_1\right)\end{aligned}$$

Knowing that $p_1 \in [0, 1]$, whatever q_1, the p_1 that maximizes the last result is $p_1 = 0$. As $p_1 + p_2 = 1$, $p_2 = 1$ and so, $p = (0, 1)$, that is, $\mathcal{BR}_A\left(q\right) = (0, 1)$. Applying the same reasoning to player B we get $\mathcal{BR}_B\left(p\right) = (0, 1)$

In this example we are in the presence of a constant Best Response function. Hence we have,

$$\begin{cases} (0,1) \in \mathcal{BR}_A\left((0,1)\right) \\ (0,1) \in \mathcal{BR}_B\left((0,1)\right) \end{cases} \Rightarrow \begin{cases} p* = (0,1) \\ q* = (0,1) \end{cases}$$

Therefore, the Nash Equilibrium of the game is (Confess, Confess).

Recalling that $R_i^A, i = 1, 2$ is the preference region of strategy i for player A, this is, the set of strategies of B to which A best response is i. So, we have R_1^A and $R_2^A \subseteq \Sigma^B$. Similarly to player B where R_1^B and $R_2^B \subseteq \Sigma^A$.

Given this and the definition on preference region given before, we are able to compute the preference regions of strategies 1 and 2 and the corresponding indifferent sets for each one of the players,

$$R_1^A = \mathcal{BR}_A^{-1}\left(e_1\right) = \varnothing, \quad R_1^B = \mathcal{BR}_B^{-1}\left(e_1\right) = \varnothing$$

$$R_2^A = \mathcal{BR}_A^{-1}\left(e_2\right) = \Sigma^B, \quad R_2^B = \mathcal{BR}_B^{-1}\left(e_2\right) = \Sigma^A$$

and $$Z_{12}^A = R_1^A \cap R_2^A = \varnothing, \quad Z_{12}^B = R_1^B \cap R_2^B = \varnothing$$

Example 2 (Matched Pennies Game)**.** Sometimes there's no pure Nash equilibrium at all. This example is one such case. In these cases we make predictions about players behaviour. Considering that they may behave randomly, we enlarge the set of strategies to include the possibility of randomization. Even in these cases, as John Nash conclude, the equilibria always exist.

Matching Pennies is a zero-sum game in that one player's gain is the other's loss. It is played between two players, A and B. Each one of them has a coin labelled with a head (H) and a tail (T). Each player should turn the coin and check if it became head or tail. Then, simultaneously, they reveal the result to each other. If the result match, this means, both are heads or both are tails, player A wins and keeps both coins. In this scenario, A gets the player B coin, so he wins $+1$ and B losses his coin, this means he gets -1. Otherwise, if the result do not match (one is head and other one is tail) player B keeps both coins. So, he gets $+1$ and player A gets -1.

We can illustrate the game in a payoff matrix, where in each cell we have the correspondent payoff for each player.

$$\begin{array}{ccc} & & \text{player B} \\ & & \begin{array}{cc} \quad H \quad & \quad T \quad \end{array} \\ \text{player A} & \begin{array}{c} H \\ \\ T \end{array} & \left(\begin{array}{cc} \begin{bmatrix} & -1 \\ +1 & \end{bmatrix} & \begin{bmatrix} & +1 \\ -1 & \end{bmatrix} \\ \begin{bmatrix} & +1 \\ -1 & \end{bmatrix} & \begin{bmatrix} & -1 \\ +1 & \end{bmatrix} \end{array}\right) \end{array}$$

The individual payoff matrices are,

$$A = \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & +1 \\ +1 & -1 \end{bmatrix}$$

Looking at the individual payoffs matrices we can verify that if player B chooses Head, A gets better choosing Head too. However, if B plays Tail, A chooses Tail as well. The same happens with player B. If A picks Head, B will choose Tail in order to maximize their own payoff. But if A plays Tail, B gets better with Head.

As we can see, in this game there's no pure strategy Nash equilibrium, since there is no pair of strategies (heads or tails) that are best responses to each other. For any pair of strategies, one of the players wants to change what they are doing. In this situation we shouldn't consider simply the strategies H and T, but also the mixed strategies. If no equilibrium exists in pure strategies, one must exist in mixed strategies. Mixed strategy is a probability distribution over two or more pure strategies. When we do this we are introducing randomized behaviour, this means that each player is not choosing a particular choice between H and T but rather is choosing the probability which he will play H or T.

Therefore, the possible strategies for both players are numbers between 0 and 1. Let us consider $p_1 \in [0,1]$ the probability of player A plays Head, so $(1-p_1) \in [0,1]$ is the probability of player A plays Tail. Similarly for player B, $q_1 \in [0,1]$ is the probability which he chooses to play Head and $(1-q_1) \in [0,1]$ is the probability which he plays Tail.

Now, each player do not have only two possible strategies but rather a set of strategies corresponding to the interval of numbers between 0 and 1. This is what we call mixed strategies.

To find the Nash Equilibrium, just as we did for the Prisoner's example, we will compute first $\mathcal{BR}_A$ and $\mathcal{BR}_B$.

$$\begin{aligned}\mathcal{BR}_A(q) &= \arg\max_p \begin{bmatrix} p_1 & 1-p_1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ 1-q_1 \end{bmatrix} = \\ &= \arg\max_p \left(2p_1\left(2q_1-1\right)-2q_1+1\right) = \arg\max_p \left(\left(2p_1-1\right)\left(2q_1-1\right)\right)\end{aligned}$$

We want to maximize $((2p_1-1)(2q_1-1))$ with respect to p_1. When $q_1 \geq \frac{1}{2}$ the expression $(2q_1-1)$ is positive and the p_1 that maximizes $\mathcal{BR}_A$ is $p_1 = 1$. On the other hand, when $q_1 \leq \frac{1}{2}$ we have $(2q_1-1) < 0$ and the maximum is reached when $p_1 = 0$. Summing up, player A chooses to play H, this means, $\mathcal{BR}_A(q) = (1,0)$, when $q_1 \geq \frac{1}{2}$ and he prefers to play T, $\mathcal{BR}_A(q) = (0,1)$, when $q_1 \leq \frac{1}{2}$. If $q_1 = \frac{1}{2}$ he is indifferent between plays H and T. In that case the best response is any $p_1 \in (0,1)$.

Analogously, we can compute $\mathcal{BR}_B(p)$. After some calculations, we get that if $p_1 \geq \frac{1}{2}$ the strategy q_1 that maximizes $\mathcal{BR}_B$ is $q_1 = 0$. This is player B prefers T instead of H and so, $\mathcal{BR}_B(p) = (0,1) = e_2$. When $p_1 \leq \frac{1}{2}$, the maximum is obtained if he chooses to play H, $q_1 = 1$, and $\mathcal{BR}_B(p) = (1,0) = e_1$. In the case where $p_1 = \frac{1}{2}$ he is indifferent between all his possible strategies. Therefore, we get

$$\mathcal{BR}_A(q) = \begin{cases} (1,0), & q_1 \geq \frac{1}{2} \\ (0,1), & q_1 \leq \frac{1}{2} \end{cases}, \quad \mathcal{BR}_B(p) = \begin{cases} (0,1), & p_1 \geq \frac{1}{2} \\ (1,0), & p_1 \leq \frac{1}{2} \end{cases}$$

Now we will compute the preference regions for each one of the players.

Preference Region of strategy 1 for player A :

$$R_1^A = \mathcal{BR}_A^{-1}(e_1) = \mathcal{BR}_A^{-1}((1,0)) = \left\{(q_1, q_2) : q_1 \geq \frac{1}{2}\right\}$$

Preference Region of strategy 2 for player A :

$$R_2^A = \mathcal{BR}_A^{-1}(e_2) = \mathcal{BR}_A^{-1}((0,1)) = \left\{(q_1, q_2) : q_1 \leq \frac{1}{2}\right\}$$

Preference Region of strategy 1 for player B :

$$R_1^B = \mathcal{BR}_B^{-1}(e_1) = \mathcal{BR}_B^{-1}((1,0)) = \left\{(p_1, p_2) : p_1 \leq \frac{1}{2}\right\}$$

Preference Region of strategy 2 for player B :

$$R_2^B = \mathcal{BR}_B^{-1}(e_2) = \mathcal{BR}_B^{-1}((0,1)) = \left\{(p_1, p_2) : p_1 \geq \frac{1}{2}\right\}$$

The indifference sets, $Z_{ii'}^A$ and $Z_{jj'}^B$ are

$$Z_{12}^A = R_1^A \cap R_2^A = \left(\frac{1}{2}, \frac{1}{2}\right) \quad \text{and} \quad Z_{12}^B = R_1^B \cap R_2^B = \left(\frac{1}{2}, \frac{1}{2}\right)$$

By Lemma 1, the game has a mixed Nash Equilibrium $\left(\frac{1}{2}, \frac{1}{2}\right)$. See Figure A.1.

Chapter 3

Fictitious Play Dynamics

In this chapter we will introduce some basic notions about dynamical system modelling the repeated or continuous play of a game, in other words, we will define the fictitious play (FP). This type of games is based on the following rule:

At each stage of the game, each player determines the (mixed) average strategy of him opponent's play up to this time and plays a (pure) best response to it.

Recall that $\Sigma = \Sigma^A \times \Sigma^B$ is the space of mixed strategy profiles where the dynamical system of fictitious play is defined and under the hypothesis that the players follow a stationary strategy distribution along entire game. Fictitious play is an algorithm seen as the dynamics beliefs of the opponent about the player's strategy distribution.

Often the fictitious play algorithm is called *myopic* learning rule. This happens because players ambition at maximize the next-round payoff looking only for the past of the play. They do not try to produce any extra strategic considerations to affect their opponent's future behaviour.

Fictitious Play has two versions: discrete-time and continuous-time. Both of them were present by Brown in 1951 (see Brown (1951)).

In the beginning, the discrete version was more famous, designed to be an algorithm for numerical approximation of Nash equilibrium in zero-sum games. This happened due to the work of Robinson in 1951 (see Robinson. (1951)) where he showed that the process converges to Nash Equilibrium in zero-sum games. To more detail about discrete-time process see also Berger (2007)

In present chapter we will defining both, discrete and continuous time versions. (see Ostrovski (2013))

3.1 Discrete-time fictitious play

As usual, consider two players, A and B. Let (A, B) be an $m \times n$ bimatrix games. The strategy set for player A is S^A and for player B, S^B. They play repeatedly at times $k \in \mathbb{N}_0$. Assume that $(x_k, y_k) \in S^A \times S^B$ denote the (pure) strategies chosen by the players at time $k \in \mathbb{N}$, with an initial condition $(x_0, y_0) \in \Sigma$.

For $k \in \mathbb{N}$, we can compute

$$p_k := \frac{1}{k}\sum_{i=0}^{k-1} x_i \in \Sigma^A \qquad \textit{and} \qquad q_k := \frac{1}{k}\sum_{i=0}^{k-1} y_i \in \Sigma^B$$

where p_k and q_k are the empirical average play through time $k-1$.

The fictitious play rule, presented in the beginning of this chapter, requires that,

$$x_k \in \mathcal{BR}_A(q_k) \cap S^A \textit{ and } y_k \in \mathcal{BR}_B(p_k) \cap S^B \tag{3.1}$$

for all times $k \in \mathbb{N}$

We can also describe p_k and q_k as the beliefs of the two players, A and B, about the strategy distribution of respective their opponent. By definition, we are able to calculate the set where p_{k+1} and q_{k+1} belong

$$\begin{aligned}
p_{k+1} &= \frac{1}{k+1}\sum_{i=0}^{k} x_i \\
&= \frac{1}{k+1}x_k + \frac{k}{k+1}\frac{1}{k}\sum_{i=0}^{k-1} x_i \\
&= \frac{1}{k+1}x_k + \frac{k}{k+1}p_k \\
&\in \frac{1}{k+1}\mathcal{BR}_A(q_k) + \frac{k}{k+1}p_k \\
&= \frac{1}{k+1}\left(\mathcal{BR}_A(q_k) + kp_k\right)
\end{aligned}$$

Equivalently,

$$q_{k+1} \in \frac{1}{k+1}\mathcal{BR}_B(p_k) + \frac{k}{k+1}q_k$$

Definition 5. For a bimatrix game (A, B) with mixed strategy space Σ, *discrete-time fictitious play* is the process $(p_k, q_k) \in \Sigma, k \geq 1$, given by the initial condition $(p_1, q_1) \in \Sigma$ and for $k \geq 1$,

$$p_{k+1} \in \frac{1}{k+1}\left(\mathcal{BR}_A\left(q_k\right) + kp_k\right) \tag{3.2}$$

$$q_{k+1} \in \frac{1}{k+1}\left(\mathcal{BR}_B\left(p_k\right) + kq_k\right) \tag{3.3}$$

Remark 1. *Geometrically, when players A and B have beliefs* $(p_k, q_k) \in R_{ij} \subset \Sigma$ *or, in other words, their best responses to the opponent's play are i and j, respectively, then* p_{k+1} *is in the line segment between* p_k *and* $e_i \in \Sigma^A$ *and the same happens to* q_{k+1}*, which will lies on the line segment between* q_k *and* $e_j \in \Sigma^B$*. So, we can say that both players' beliefs* (p_k, q_k) *move towards their currently preferred pure strategy, with step size decreasing with time k.*

Remark 2. *Notice that,*

$$\begin{aligned}
p_{k+1} - p_k &= \frac{1}{k+1}\sum_{i=0}^{k} x_i - p_k \\
&= \frac{1}{k+1}x_k + \frac{k}{k+1}\frac{1}{k}\sum_{i=0}^{k-1} x_i - p_k \\
&= \frac{1}{k+1}x_k + \frac{k}{k+1}p_k - p_k \\
&\in \frac{1}{k+1}\mathcal{BR}_A\left(q_k\right) + \frac{k}{k+1}p_k - p_k \\
&= \frac{1}{k+1}\left(\mathcal{BR}_A\left(q_k\right) + kp_k - (k+1)\,p_k\right) \\
&= \frac{1}{k+1}\left(\mathcal{BR}_A\left(q_k\right) - p_k\right)
\end{aligned}$$

and,

$$q_{k+1} - q_k \in \frac{1}{k+1}\left(\mathcal{BR}_B\left(p_k\right) - q_k\right)$$

In order that $\|p_{k+1} - p_k\|$ *and* $\|q_{k+1} - q_k\|$ *are bounded by* $\frac{\sqrt{2}}{(k+1)}$*. This means that the step size of fictitious play decreases like* $\frac{1}{k}$*. This express the fact that new data about the opponent has decreasing impact as k grow.*

3.2 Continuous-time fictitious play

In this section we will assume a continuous-time process, instead of a game that are played again and again.

A continuous-time fictitious play is seen as a dynamical system where players are presumed to continuously play a given bimatrix game by choosing the best response to the average of their respective opponent's past play at each time $t > 0$ (see Hofbauer & Sigmund (1998)).

Formalizing this notion mathematically we have that

$$p(t) = \frac{1}{t}\int_0^t \mathcal{BR}_A(q(s))\,ds \Leftrightarrow tp(t) = \int_0^t \mathcal{BR}_A(q(s))\,ds$$

Deriving both sides in order to t, we get

$$p(t) + tp'(t) = \mathcal{BR}_A(q(t)) \Leftrightarrow p'(t) = \frac{1}{t}\left(\mathcal{BR}_A(q(t)) - p(t)\right)$$

Given this and doing a parallel with discrete-time studied in last section, we give the following definition.

Definition 6. For a bimatrix game (A, B) with mixed strategy space Σ, *continuous-time fictitious play* is the process $(p(t), q(t)) \in \Sigma, t \geq t_0 > 0$, given by the differential inclusion

$$\dot{p}(t) \in \frac{1}{t}\left(\mathcal{BR}_A(q(t)) - p(t)\right) \text{ and } \dot{q}(t) \in \frac{1}{t}\left(\mathcal{BR}_B(p(t)) - q(t)\right)$$

considering some initial condition $(p(t_0), q(t_0)) = (p_0, q_0) \in \Sigma$.

Remark 3. *(1) Based on last definition we note that fictitious play is a differential inclusion. We cannot ensure the uniqueness of solutions. However, by general theory, we can ensure that solutions exist for all initial conditions. This conclusion is due to the fact that $\mathcal{BR}_A$ and $\mathcal{BR}_B$ are upper semi-continuous correspondences with closed and convex values (faces of Σ^A and Σ^B) (see Aubin & Cellina. (1948))*

(2) We can even conclude that Nash equilibrium are precisely the equilibrium of fictitious play dynamics FP. Particularly, if an orbit of FP converges to a single point, that point is a Nash equilibrium.

(3) As we have seen before, in discrete-time case, $\mathcal{BR}_A$ and $\mathcal{BR}_B$ are piecewise constant on the convex sets $R_j^B \subseteq \Sigma^A$ and $R_i^A \subseteq \Sigma^B$, respectively.

Based on this fact we can check that orbits are locally straight line segments going for vertices of Σ. They just alter their trajectory direction when they hit on an indifference set, in order words, when they change the block R_{ij}. (next figure is an example of what happen in case 2×2 games)

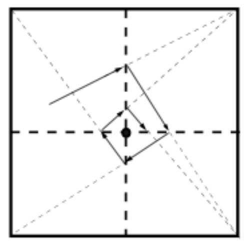

Figure 3.1: Trajectory over time

(4) As we see in last definition, Fictitious Play is time-dependent. However, if we make a change of variable in time, we can turns it into an autonomous system. We can assume that without loss of generality because the time dependence is related with the slowing down of the motion as time progresses and no related with the actual shape of the trajectories.
So, considering that $t = e^{\tau}$ and so $\tilde{p}(\tau) = p(e^{\tau})$ and $\tilde{q}(\tau) = q(e^{\tau})$ we get,

$$\frac{\partial \tilde{p}}{\partial \tau} = \frac{\partial p}{\partial t}(e^{\tau}) \frac{\partial (e^{\tau})}{\partial \tau} = p'(e^{\tau}) e^{\tau} = \frac{1}{e^{\tau}} \left(\mathcal{BR}_A (q(e^{\tau})) - p(e^{\tau}) \right) e^{\tau} =$$
$$= \mathcal{BR}_A (q(e^{\tau})) - p(e^{\tau})$$

Therefore,

$$\tilde{p}'(\tau) = \mathcal{BR}_A (\tilde{q}(\tau)) - \tilde{p}(\tau)$$

Now we can think about FP as an autonomous system. This will simplify certain arguments. We can give, now a formal definition for this system (see Hofbauer & Sigmund (1998)).

Definition 7. For a bimatrix game (A, B) with mixed strategy space Σ, *best response dynamics* (BR) is the process $(p(t), q(t)) \in \Sigma$, $t \geq 0$, given by the differential inclusion,

$$\dot{p}(t) \in \mathcal{BR}_A (q(t)) - p(t) \quad \text{and} \quad \dot{q}(t) \in \mathcal{BR}_B (p(t)) - q(t)$$

with some initial condition $(p(0), q(0)) = (p, q) \in \Sigma$.

We will denote by $x(t)$, $x : [0, \infty) \to \Sigma^A$ the strategy played by player A and $y(t)$, $y : [0, \infty) \to \Sigma^B$ by player B, at time $t > 0$.

Example 3 (Prisoner's Dilemma cont.)**.** Let us apply these concepts to the example of the prisoners dilemma presented in the last chapter. Now, we are supposing that players want their strategies evolve over the time. Notice that

$$\Sigma^A = \Sigma^B = \left\{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 + x_2 = 1\right\}$$

According to the definition of Fictitious Play, strategies p and q evolve according to the equations

$$\begin{cases} \dot{p} = \mathcal{BR}_A(q) - p \\ \dot{q} = \mathcal{BR}_B(p) - q \end{cases}$$

Implement this in Prisoner's Dilemma we get

$$\begin{cases} \dot{p} = (0,1) - (p_1, p_2) \\ \dot{q} = (0,1) - (q_1, q_2) \end{cases} \Leftrightarrow \begin{cases} (\dot{p_1}, \dot{p_2}) = (-p_1, 1 - p_2) \\ (\dot{q_1}, \dot{q_2}) = (-q_1, 1 - q_2) \end{cases}$$

Solving this system of ODEs we obtain the following results

$$\begin{cases} p_1(t) = e^{-t} p_1(0) \\ p_2(t) = 1 - e^{-t} p_1(0) \\ q_1(t) = e^{-t} q_1(0) \\ q_2(t) = 1 - e^{-t} q_1(0) \end{cases}$$

We can even check that over time the solution will moves towards the best response

$$\lim_{t\to\infty} p(t) = \lim_{t\to\infty} (p_1(t), p_2(t)) = (0,1), \ \lim_{t\to\infty} q(t) = \lim_{t\to\infty} (q_1(t), q_2(t)) = (0,1)$$

Dynamics of ODE can be sketch in the following phase portrait. (see Figure A.2). It is generated by a product of two intervals $\Delta^1_A \times \Delta^1_B = [0,1] \times [0,1]$, each one correspondent to strategies interval for p_1 and q_1, respectively. Note that strategies p_2 and q_2 can be obtained at the expense of p_1 and q_1, respectively.

3.3 Uniqueness of fictitious play flow

As we said before, whatever it is the initial condition $(p_0, q_0) \in \Sigma$, the differential inclusion assumes solutions. The guarantee of uniqueness and continuity of the flow induced by this system is not so true. It just holds for generic bimatrix games and not for all.

The problem happens when the solution curves crosses any one of the indifference sets $Z^A = \bigcup_{i,i'} Z^A_{ii'}$ or $Z^B = \bigcup_{j,j'} Z^B_{jj'}$. In these sets, the right-hand sides are multi-valued and that can provide non-uniqueness of solutions. Locally we do not have these problems. In fact, in the interior of the convex sets $R_{ij} \subseteq \Sigma$, the solutions of FP are unique and continuous and the segments are straight lines. To work around this problem, all of $Z^A_{ii'}$ and $Z^B_{jj'}$ are required to be codimension-one planes and that the flow crosses them transversally. It follows a proposition.

Proposition 1. *(see Sparrow (2008)) Let (A, B) be an $m \times n$ bimatrix game. Denote $Z^* = Z^B \times Z^A$ the set where each of the players is indifferent between at least two strategies. Assume that for all $(p, q) \in \Sigma \setminus Z^*$, if $p \in Z^B$ and $q \notin Z^A$ so that, say, $\mathcal{BR}_A(q) = e_k$, then e_k is not parallel to the plane $Z^B \subset \Sigma^A$ at the point p, and similarly for the roles of p and q reversed. Then FP defines a continuous flow on $\Sigma \setminus Z^*$.*

After a more detailed analysis about the set Z^* it is possible to obtain a stronger result. To get more detail about this see Sparrow (2008).

Proposition 2. *Assume the bimatrix game (A, B) satisfies the hypotheses of last proposition and additionally assume that A and B have maximal rank. Then the flow on the interior of the sets R_{ij} has a unique continuous extension everywhere (on Σ), except possibly points in the subset of Z^* where one of the players is indifferent between at least three of his strategies. This remaining set has a codimension three.*

Chapter 4

Analysis of 2×2 Fictitious Play

One of the earliest conclusions in game theory was that in any 2×2 bimatrix game all solutions of Fictitious Play converge to a point, the Nash equilibrium point. Miyasawa (see Miyasawa (1961)) give us the first proof of this result. Later Metrick and Polack (see Metrick & Polak (1994)) present a more conceptual and geometric proof.

Theorem 1 (Miyasawa, Metrick and Polak)**.** *For any 2×2 bimatrix game (A, B), any orbit $\{(p(t), q(t)), t \geq t_0\}$ of FP with initial condition $(p(t_0), q(t_0)) = (p_0, q_0) \in \Sigma$ for some $t_0 \geq 0$ converges to a Nash equilibrium $(p^*, q^*) \in \Sigma$ as $t \to \infty$.*

In this chapter we will analyse all the possible types of phase portraits arising in 2×2 fictitious play.

4.1 Indifference sets

As we already said, for generic values of p and q, $\mathcal{BR}_A$ and $\mathcal{BR}_B$ are in general a pure strategy. But in fact, that is not always true. As we have seen, when we are looking at the indifference sets the best response consists in more than one pure strategy. In these sets we face with $e_1Aq = e_2Aq$ or $pBe_1 = pBe_2$ for player A and B, respectively. This means that both of pure strategies, e_1 and e_2, provide the same utility to the players.

Therefore, also any convex combination between e_1 and e_2 gives to A the same utility, and the same is valid for player B.

4.1.1 Identification of the indifference sets

In the present section we will show you how to explicitly identify indifference sets, Z_{ij}^A and Z_{ij}^B. For better understand the dynamics of this we will first present you a generic 2×2 case.

For player A, consider that a_{ij} is his pay-off when he chooses the strategy i given that player B played j. The pay-off matrix for player A is

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Remember that, $q_1 + q_2 = 1$, $e_1 = (0,1)$ and $e_2 = (1,0)$. Developing the definition of indifference set we get :

$$\begin{aligned} & e_1 A q = e_2 A q \Leftrightarrow \\ & \Leftrightarrow \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \Leftrightarrow \\ & \Leftrightarrow q_1 = \frac{(a_{22} - a_{12})}{(a_{11} - a_{21})\,(a_{22} - a_{12})} \Leftrightarrow \\ & \Leftrightarrow q_1 = \frac{1}{\frac{(a_{11} - a_{21})}{(a_{22} - a_{12})} + 1} \end{aligned}$$

Looking at this result we are able to say that, in the case 2×2 fictitious games, the indifference set is just a point. Denote this point by q_1^*. We know that $q_1 \in (0,1)$, so even q_1^*. In order for $q_1^* \in (0,1)$ we need to ensure that

$$\sigma_A = \frac{(a_{11} - a_{21})}{(a_{22} - a_{12})} \geq 0$$

For player B, using the same principles we will compute Z_{ij}^B. Let the pay-off matrix for player B be

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Just as we did for player A we will now develop the equality given in indifference set of B $pBe_1 = pBe_2$.

$$
\begin{aligned}
& pBe_1 = pBe_2 \Leftrightarrow \\
& \Leftrightarrow \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Leftrightarrow \\
& \Leftrightarrow \begin{bmatrix} p_1 b_{11} + p_2 b_{21} & p_1 b_{12} + p_2 b_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} p_1 b_{11} + p_2 b_{21} & p_1 b_{12} + p_2 b_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Leftrightarrow \\
& \Leftrightarrow p_1 = \frac{1}{\frac{(b_{11}-b_{12})}{(b_{22}-b_{21})} + 1}
\end{aligned}
$$

Using the same notations that we use for player A, let us denote the point p_1 of indifference set by p_1^*. We know that $p_1^* \in (0,1)$ so we have to certify that

$$\sigma_B = \frac{(b_{11} - b_{12})}{(b_{22} - b_{21})} \geq 0$$

Definition 8. We say that player A is independent from player B moves if $Z_{12}^A = \varnothing$. Similarly, B is independent from A choices if $Z_{12}^B = \varnothing$

In other words, saying that A is independent from B is the same as saying that player A does not change his strategy regardless of B's choice. Given this, player A has no indifference sets.

Lemma 2. *player A is not independent from player B moves, if and only if,*

$$\sigma_A = \frac{(a_{11} - a_{21})}{(a_{22} - a_{12})} > 0$$

This implies that $q_1^* \in (0,1)$ and so Z_{12}^A is not empty.

Analogously, player B is not independent from A if and only if,

$$\sigma_B = \frac{(b_{11} - b_{12})}{(b_{22} - b_{21})} \geq 0$$

Which means that $p_1^* \in (0,1)$ and so Z_{12}^B is not empty.

Remark 4. *Moreover, if $(q_1^*, q_2^*) \in Z_{12}^A$ then*

$$q_1^* = \frac{1}{\sigma_A + 1}$$

Equivalently, if $(p_1^*, p_2^*) \in Z_{12}^B$ then

$$p_1^* = \frac{1}{\sigma_B + 1}$$

Example 4 (Simple example)**.** Consider player A pay-off matrix,

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Lets compute the value of

$$\sigma_A = \frac{(a_{11} - a_{21})}{(a_{22} - a_{12})} = \frac{2-1}{2-1} = 1 > 0$$

Consequently, A is not independent from B and $q_1^* = \frac{1}{1+1} = \frac{1}{2}$. We are also able to compute the value of $q_2^* = 1 - q_1^* = 1 - \frac{1}{2} = \frac{1}{2}$. Thus, as we said before, the indifference set corresponds to a point $Z_{12}^A = \left\{(\frac{1}{2}, \frac{1}{2})\right\}$.

The point $q^* = (\frac{1}{2}, \frac{1}{2})$ is the strategy of B for which A is completely indifferent to play e_1 or e_2. We can see this graphic representation in Figure A.3.

If player B chooses a strategy q located in Z_1 we get $\mathcal{BR}_A(q) = e_1$. On the other hand, if B q is in Z_2, $\mathcal{BR}_A(q) = e_2$. As we had already conclude, in the graphic we can certify that A is not independent from B's choices since he changes his best strategy depending on B. His strategy depends where q is posted.

In the previous paragraph we assume that if $q \in Z_1$ A chooses strategy e_1 and if $q \in Z_2$ A prefer e_2. But how can we know that ?

First we assume that player B plays e_2 this is, $q = (0, 1)$. Now we will compute $\mathcal{BR}_A(e_2)$ to check which strategy gives to A the largest pay-off in response to B choice.

- If A plays e_1 :

$$e_1 A q = e_1 \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} e_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1$$

- If A plays e_2 :

$$e_2 A q = e_2 \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} e_2 = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2$$

As $2 > 1$, when B picks e_2, player A gets higher pay-off if he chooses e_2 too. Since $e_2 \in Z_2$ we can assume that in that region A will answer with e_2 in order to maximize his utility.

Using the same reasoning, we can check what happens when player B picks e_1. We will reach the conclusion that player A will prefer e_1 in response to e_1 played by B. So, in region Z_1 player A will be more satisfied if he chooses e_1.

Now, we will see, in the general case, how can we know the best response of A for a given pure strategy played by B.

First we assume that B plays $q = e_2 = (0, 1)$. Let's figure out which is the best response of player A.

$$\begin{cases} e_1 A e_2 = e_1 \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} e_2 = a_{12}, & A \text{ plays } e_1 \\ e_2 A e_2 = e_2 \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} e_2 = a_{22}, & A \text{ plays } e_2 \end{cases}$$

Therefore, we just need to compare the values of a_{12} and a_{22} in order to know the preferences of player A in response to player B choice e_2.

- If $a_{22} > a_{12}$ we get

  $$\mathcal{BR}_A(q) = \begin{cases} e_2, & q_2 > q_2^* \\ e_1, & q_2 < q_2^* \end{cases} = \begin{cases} e_2, & q_1 < q_1^* \\ e_1, & q_1 > q_1^* \end{cases}$$

- If $a_{12} > a_{22}$, we obtain

  $$\mathcal{BR}_A(q) = \begin{cases} e_1, & q_2 > q_2^* \\ e_2, & q_2 < q_2^* \end{cases} = \begin{cases} e_1, & q_1 < q_1^* \\ e_2, & q_1 > q_1^* \end{cases}$$

Analogously, assuming that player B play e_1, this means that, $q = (1, 0)$ we get

$$\begin{cases} e_1 A e_1 = e_1 \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} e_1 = a_{11}, & A \text{ plays } e_1 \\ e_2 A e_1 = e_2 \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} e_1 = a_{21}, & A \text{ plays } e_2 \end{cases}$$

In this case we just have to confront the values of a_{11} and a_{21} to find out the best option for player A.

If $a_{11} > a_{21}$ player A prefer to play e_1. In the other hand, if $a_{11} < a_{21}$ player A gets more utility if he chooses e_2. Summarize,

- When $a_{11} > a_{21}$

  $$\mathcal{BR}_A(q) = \begin{cases} e_1, & q_1 > q_1^* \\ e_2, & q_1 < q_1^* \end{cases}$$

- In the other hand, when $a_{11} < a_{21}$

  $$\mathcal{BR}_A(q) = \begin{cases} e_1, & q_1 < q_1^* \\ e_2, & q_1 > q_1^* \end{cases}$$

Proposition 3. *We can summing up these 4 cases.*

$$\mathcal{BR}_A(q) = \begin{cases} e_1, & (q_1 > q_1^* \wedge a_{11} > a_{21}) \vee (q_1 < q_1^* \wedge a_{12} > a_{22}) \\ e_2, & (q_1 > q_1^* \wedge a_{11} < a_{21}) \vee (q_1 < q_1^* \wedge a_{12} < a_{22}) \end{cases}$$

Doing the same analysis for player B, this is, assuming that player A plays his pure strategies, $p = e_1$ or $p = e_2$, we will compute the best responses of B.

We will apply exactly the same procedure as we did before. First assuming that A plays $p = e_2 = (0, 1)$.

$$\begin{cases} e_2 B e_1 = e_2 \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} e_1 = b_{21}, & B \text{ plays } e_1 \\ e_2 B e_2 = e_2 \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} e_2 = b_{22}, & B \text{ plays } e_2 \end{cases}$$

Concluding we just have to contrast the values of b_{21} and b_{22} to get the best response of B when A plays e_2. If $b_{21} > b_{22}$ B gets more utility if he plays e_1. On the other hand, when $b_{21} < b_{22}$ B prefers the strategy e_2.

Consider now that player A plays $p = e_1 = (1, 0)$.

$$\begin{cases} e_1 B e_1 = e_1 \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} e_1 = b_{11}, & B \text{ plays } e_1 \\ e_1 B e_2 = e_1 \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} e_2 = b_{12}, & B \text{ plays } e_2 \end{cases}$$

In this case we will compare the values of b_{11} and b_{12}. If $b_{11} > b_{12}$ player B will take the strategy e_1. However, if $b_{11} < b_{12}$ he opts by e_2.

Just as we do for player A let's summing up these 4 results in a proposition.

Proposition 4.

$$\mathcal{BR}_B(p) = \begin{cases} e_1, & (p_1 > p_1^* \wedge b_{11} > b_{12}) \vee (p_1 < p_1^* \wedge b_{21} > b_{22}) \\ e_2, & (p_1 > p_1^* \wedge b_{11} < b_{12}) \vee (p_1 < p_1^* \wedge b_{21} < b_{22}) \end{cases}$$

Now let us study and apply these concepts in another example of 2×2 fictitious play.

4.1.2 Examples

Example 5 (A not independent from B but B independent from A)**.** Consider the pay-off matrices for player A and B respectively,

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

In order to know the indifference sets, we will compute the values of σ_A and σ_B,

$$\sigma_A = \frac{(a_{11} - a_{21})}{(a_{22} - a_{12})} = \frac{3-1}{2-1} = 2 > 0$$

and

$$\sigma_B = \frac{(b_{11} - b_{12})}{(b_{22} - b_{21})} = \frac{0-1}{2-1} = -1 < 0$$

For player A we have that $Z_{12}^A \neq \varnothing$ and $q_1^* = \frac{1}{2+1} = \frac{1}{3}$, $q_2^* = 1 - q_1^* = \frac{2}{3}$. And so $Z_{12}^A = \left\{ \left(\frac{1}{3}, \frac{2}{3} \right) \right\}$

Although for player B, $Z_{12}^B = \varnothing$. This means that player B is independent from A and therefore he has only one strategy that maximizes his pay-off whatever the A plays.

In this case our scenario is,
- A is not independent from B
- B is independent from A

The next step is to analyse the phase portrait for this scenario where we have a strategy q^* of B that divides the best responses of A, but we don't have the strategy p^*.

To player A we have $a_{22} = 2$ and $a_{12} = 1$, so $a_{22} > a_{12}$, therefore (See Figure A.4)

$$\mathcal{BR}_A(q) = \begin{cases} e_2, & q_1 \leq q_1^* \\ e_1, & \text{otherwise} \end{cases} = \begin{cases} e_2, & q_1 \leq \frac{1}{3} \\ e_1, & \text{otherwise} \end{cases}$$

We already noticed the behaviour of player A. Now let us find the deportment of player B. As we saw, player B has just one best response whatever p is. Therefore, in order to know which one of pure strategies B prefer, we will assume that player A played e_1 and verify how B reacts, if he gets higher pay-off using e_1 or e_2.

Following previous discussion, since $b_{12} > b_{11}$, we get $\mathcal{BR}_B(p) = e_2$.

Next we will determine Nash equilibrium, $(\overline{p},\overline{q}) : \overline{p} \in \mathcal{BR}_A(\overline{q})$ and $\overline{q} \in \mathcal{BR}_B(\overline{p})$. Since B is independent from A, it is understandable that $\mathcal{BR}_B(\overline{p}) = e_2$, hence $\overline{q} = e_2$ for all values of p. Taking this into account, we are able to identify the $\mathcal{BR}_A(\overline{q})$,

$$\mathcal{BR}_A(\overline{q}) = \mathcal{BR}_A(e_2) = \mathcal{BR}_A((0,1)) = e_2, \quad \text{since } q_1 = 0$$

Summarizing, $(\overline{p},\overline{q}) = (e_2, e_2)$ and $\overline{p} = (0,1)$ and $\overline{q} = (0,1)$.

We can include this result in the Figure A.4, see Figure A.5.

The next step is to sketch the phase portrait of the associated fictitious play. Remember that p_2 and q_2 can be obtained from p_1 and q_1. Until here, our analysis has been done in terms of p_1 and q_1 and for that reason we will develop the next computations in these coordinates too. As we remember,

$$\begin{cases} \dot{p}_1 & = & \begin{cases} -p_1, & q_1 \leq \frac{1}{3} \\ 1 - p_1, & \text{otherwise} \end{cases} \\ \dot{q}_1 & = & -q_1 \end{cases}$$

We can automatically solve the second equation,

$$\dot{q}_1 = -q_1 \quad \Leftrightarrow \quad q_1(t) = e^{-t} q_1(0)$$

which is an asymptotically stable equilibrium. Whatever the initial condition $q_1(0) \in (0,1)$, q_1 will always converges to zero. (See Figure A.6)

Conversely, in relation to p_1 dynamics we need to take into account the initial condition. We have two options for that,

$$(1) \quad q_1(0) \in \left[\frac{1}{3}, 1\right] \quad \text{and} \quad (2) \quad q_1(0) \in \left[0, \frac{1}{3}\right]$$

When the initial condition is (1) we are in the branch $\dot{p}_1 = 1 - p_1$. In this case, the trajectory has a tendency to converge in a straight line to one. So in the equilibrium we have $p_1 \to 1$ and $q_1 \to 0$. When the first condition is localized in the button region, (2), we have $\dot{p}_1 = -p_1$ the trajectory will converges to zero. And so we have, $p_1 \to 0$ and $q_1 \to 0$. We are able now to illustrate each situation graphically, see Figure A.7. Then we can join this two situations in only one graphic, see Figure A.8.

When the trajectory crosses the boundary region, in this specific case $q_1 = \frac{1}{3}$, changes its direction starting to converge to the equilibrium point $(p_1, q_1) = (0,0)$, which is the Nash equilibrium.

In the last example we have analysed the scenario where A is not independent from B but, where's B is independent from A. However this is not the only possible case. Actually, we have four possible cases,

1. A is independent from B and B is independent from A
2. A is independent from B but B is not independent from A
3. A is not independent from B but B is independent from A
4. A is not independent from B and B is not independent from A

Just as we did for case 3., we will now present three more examples for each one of the remaining cases.

Example 6 (A independent from B and B independent from A)**.** Prisoner's dilemma, the example that we have been developing throughout this work is an example of this first case. Remember the pay-off matrices for player A and B,

$$A = \begin{bmatrix} -1 & -3 \\ 0 & -2 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 0 \\ -3 & -2 \end{bmatrix}$$

Computing the values of σ_A and σ_B :

$$\sigma_A = \frac{(a_{11} - a_{21})}{(a_{22} - a_{12})} = \frac{-1-0}{-2+3} = -1 < 0$$

$$\sigma_B = \frac{(b_{11} - b_{12})}{(b_{22} - b_{21})} = \frac{-1-0}{-2+3} = -1 < 0$$

As we can see, both, Z_{12}^A and Z_{12}^B are an empty sets, so either player A is independent of player B as player B is independent of player A. In other words, regardless of the opponent's move, both players have a unique strategy that maximizes their own pay-off.

Remember that in this specific example we have already computed the Nash equilibrium as well as the face portrait. See Figure A.2.

Until now, we had already seen first and third cases. About the second one we do not see as fundamental to develop an example here. The main idea is the same that we used for the third case. In this case, we just need to take into account that they are 'symmetric' to each other and that in the second case we will get p^* as a vertical boundary region instead of q^* horizontal boundary region, obtained in example 5.

With respect to the forth case let us bring back the example of Matched Pennies.

Example 7 (A not independent from B and B not independent from A)**.** Remember the pay-off matrices of Matched Pennies game for each one of the players,

$$A = \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & +1 \\ +1 & -1 \end{bmatrix}$$

Find the value of σ_A and σ_B,

$$\sigma_A = \frac{(a_{11} - a_{21})}{(a_{22} - a_{12})} = \frac{1 - (-1)}{1 - (-1)} = 1 > 0$$

$$\sigma_B = \frac{(b_{11} - b_{12})}{(b_{22} - b_{21})} = \frac{(-1) - 1}{(-1) - 1} = 1 > 0$$

So, we are in a presence of case four. Where both, player A and player B, depend on the opponent's moves.

In this case we have $Z_{12}^A \neq \varnothing$ and $Z_{12}^B \neq \varnothing$. Thus, we need to compute q^* and p^*,

$$q_1^* = \frac{1}{1+1} = \frac{1}{2} \quad ; \quad q_2^* = \frac{1}{2} \quad \Rightarrow Z_{12}^A = \left(\frac{1}{2}, \frac{1}{2}\right)$$

$$p_1^* = \frac{1}{1+1} = \frac{1}{2} \quad ; \quad p_2^* = \frac{1}{2} \quad \Rightarrow Z_{12}^B = \left(\frac{1}{2}, \frac{1}{2}\right)$$

So we have two strategies, q^* and p^* the first one divides the best responses of A into two parts and the second one divides the best responses of B also into two parts. We just already computed the Nash equilibrium for this example (check example 2). Just to remember we got

$$\mathcal{BR}_A(q) = \begin{cases} (1,0), & q_1 \geq \frac{1}{2} \\ (0,1), & q_1 \leq \frac{1}{2} \end{cases} \quad , \quad \mathcal{BR}_B(p) = \begin{cases} (0,1), & p_1 \geq \frac{1}{2} \\ (1,0), & p_1 \leq \frac{1}{2} \end{cases}$$

And the Nash equilibrium found was $(\overline{p_1}, \overline{q_1}) = \left(\frac{1}{2}, \frac{1}{2}\right)$

Let us now check the phase portrait.

$$\dot{p_1} = \begin{cases} 1 - p_1, & q_1 \geq \frac{1}{2} \\ -p_1, & q_1 \leq \frac{1}{2} \end{cases}, \quad \dot{q_1} = \begin{cases} -q_1, & p_1 \geq \frac{1}{2} \\ 1 - q_1, & p_1 \leq \frac{1}{2} \end{cases}$$

There are four possible of combinations between initial conditions of $p_1(0)$ and $q_1(0)$. See Figure A.9.

1. $p_1(0) \leq \frac{1}{2} \quad \wedge \quad q_1(0) \leq \frac{1}{2}$: with this combination of initial conditions we have $\dot{p_1} = -p_1$ and $\dot{q_1} = 1 - q_1$. Over time the trajectory will converge to $\dot{p_1} \to 0$ and $\dot{q_1} \to 1$, this is the point $(p_1, q_1) = (0, 1)$.

2. $p_1(0) \leq \frac{1}{2} \quad \wedge \quad q_1(0) \geq \frac{1}{2}$: here we have $\dot{p_1} = 1 - p_1$ and $\dot{q_1} = 1 - q_1$. Over time they will tend to $\dot{p_1} \to 1$ and $\dot{q_1} \to 1$ this is the point $(p_1, q_1) = (1, 1)$.

3. $p_1(0) \geq \frac{1}{2} \quad \wedge \quad q_1(0) \geq \frac{1}{2}$: in this case we have $\dot{p_1} = 1 - p_1$ and $\dot{q_1} = -q_1$. As time progresses the trajectory will run to $\dot{p_1} \to 1$ and $\dot{q_1} \to 0$ this means the point $(p_1, q_1) = (1, 0)$.

4. $p_1(0) \geq \frac{1}{2} \quad \wedge \quad q_1(0) \leq \frac{1}{2}$: the last combination is $\dot{p_1} = -p_1$ and $\dot{q_1} = -q_1$. Here the path will evolve over time to $\dot{p_1} \to 0$ and $\dot{q_1} \to 0$, This is, $(p_1, q_1) = (0, 0)$.

4.1.3 Combinatorially distinct types

In accordance with the examples of last section we are now to show that fictitious play converges to Nash equilibrium (as claimed by Theorem 1) and the phase space belongs to a finite family of combinatorial distinct types. In this subsection we will synthesize all these types. Before we start, to simplify the notation, let, $p, q \in [0, 1]$ denote the probability of player A and B, respectively.

As we saw in the last section, there is $(p^*, q^*) \in \mathbb{R}^2$, such that fictitious play can be divided in four scenarios:

$$\text{(A)} \qquad \dot{p} = \begin{cases} 1, & q \geq q^* \\ 0, & q \leq q^* \end{cases} - p \quad \text{and} \quad \dot{q} = \begin{cases} 1, & p \geq p^* \\ 0, & p \leq p^* \end{cases} - q$$

$$\text{(B)} \qquad \dot{p} = \begin{cases} 1, & q \leq q^* \\ 0, & q \geq q^* \end{cases} - p \quad \text{and} \quad \dot{q} = \begin{cases} 1, & p \geq p^* \\ 0, & p \leq p^* \end{cases} - q$$

$$\text{(C)} \qquad \dot{p} = \begin{cases} 1, & q \geq q^* \\ 0, & q \leq q^* \end{cases} - p \quad \text{and} \quad \dot{q} = \begin{cases} 1, & p \leq p^* \\ 0, & p \geq p^* \end{cases} - q$$

$$\text{(D)} \qquad \dot{p} = \begin{cases} 1, & q \leq q^* \\ 0, & q \geq q^* \end{cases} - p \quad \text{and} \quad \dot{q} = \begin{cases} 1, & p \leq p^* \\ 0, & p \geq p^* \end{cases} - q$$

For each one of these four scenarios we will see what happen with the convergence when we change the values of p^* and q^*. Changing these values we are changing the independence relationship between players and also changing the value of the Nash equilibrium. We found nine different combinations for p^* and q^*. Let us see what happen for the case (A).

1. $p^* \leq 0$ and $q^* \geq 1$ we get $\begin{cases} \dot{p} = -p \\ \dot{q} = 1 - q \end{cases}$ In this case we have that p is always greater or equal than p^* and q is always less or equal than q^*. So, over time the trajectory will converge to the Nash equilibrium point which is $(p, q) = (0, 1)$. We can see in the figure A.10 (1).

2. $p^*, q^* \geq 1$ we get $\begin{cases} \dot{p} = -p \\ \dot{q} = -q \end{cases}$ Here, as in the previous case, q is always less or equal than q^*, but in contrast with the last case p is also always less or equal than p^*. Graphically, Figure A.10 (2), we can see that, over time the trajectory will converge to the Nash equilibrium point $(p, q) = (0, 0)$.

3. $p^* \geq 1$ and $q^* \leq 0$ we get $\begin{cases} \dot{p} = 1 - p \\ \dot{q} = -q \end{cases}$. In this third case $p \leq p^*$ and $q \geq q^*$ so, over time, it will converge to the point $(p, q) = (1, 0)$ which is the Nash equilibrium. See Figure A.10 (3).

4. $p^*, q^* \leq 0$ we get $\begin{cases} \dot{p} = 1 - p \\ \dot{q} = 1 - q \end{cases}$ because p and q are always greater than p^* and q^*, respectively, over time it will converge to the point $(p, q) = (1, 1)$ which is the Nash equilibrium. See Figure A.10 (4).

5. $0 \leq q^* \leq 1$ and $p^* \leq 0$ we get $\dot{q} = 1 - q$ and so q will always converge to 1. The convergence of p will depends on initial condition. However, as time goes by, it will always converge to the Nash equilibrium point, which is $(p, q) = (1, 1)$. We can check this in figure A.10 (5).

6. $0 \leq q^* \leq 1$ and $p^* \geq 1$, we get $\dot{q} = -q$ and so q will always converge to 0. The convergence of p will also depends on initial condition. Despite this, graphically we can prove that over time it will converge to the Nash equilibrium $(p, q) = (0, 0)$ whatever it is the initial condition. Check the graph (6) in figure A.10.

7. $0 \leq p^* \leq 1$ and $q^* \leq 0$ we get $\dot{p} = 1 - p$ and so p will always converge to 1. The convergence of q will change with the local of the initial condition but as time progress it will always converge to Nash point,$(p, q) = (1, 1)$. See Figure A.10 (7).

8. $0 \leq p^* \leq 1$ and $q^* \geq 1$ here we get $\dot{p} = -p$ and so p will always converge to 0. The convergence of q will change with the local of the initial condition. Once, graphically we check the convergence to the Nash equilibrium point, $(p, q) = (0, 0)$. See Figure A.10 (8).

9. $0 \leq q^* \leq 1$ and $0 \leq p^* \leq 1$ in this last case the convergence will depends on the initial condition of both variables. If $p \leq p^*$ and $q \leq q^*$ we are in the first quadrant and the the trajectory will converge to the point $(p, q) = (0, 0)$. If we are in the second quadrant, this is $p \geq p^*$ and $q \leq q^*$, the trajectory will converge to $(p, q) = (0, 1)$ and go on. In the graphic (9) in figure A.10 we represent the three possible Nash equilibrium. We can check that the EN1 and EN3 are pure and stable. But EN2 are mixed and unstable.

We can use the same reasoning for the remaining three cases. We do not see as necessary to show all of them here because the idea is exactly the same used in case (A). However, we will develop the case 9,

$$0 \leq q^* \leq 1 \quad \text{and} \quad 0 \leq p^* \leq 1$$

for each one of the scenarios, since it is a compilation of all the other eight cases where none of the players is independent from the other.

(B) Here the trajectory is like an spiral and it will converge to the Nash equilibrium which is $(p, q) = (p^*, q^*)$. This equilibrium is mixed and stable. Check Figure A.11.

(C) In this case the trajectory is symmetrical to scenario (B). The Nash equilibrium is either the mixed point, $(p, q) = (p^*, q^*)$ and it is also stable. As we saw before, the Matched Pennies example is included in this scenario. See Figure A.12.

(D) Lastly, this scenario is the symmetric of scenario seen in case (A). There are two pure and stable equilibrium (EN1 and EN3) and one mixed and unstable equilibrium, EN2. See Figure A.13.

Chapter 5

Conclusions and Further Research

In this dissertation we have studied a specific case of evolutionary games, the 2×2 fictitious play where we have two players both with two strategies.

In case of zero-sum games, fictitious play always converge to a set of Nash equilibria. In our specific case study it was guaranteed, by Theorem 1, that the trajectories of the system always converge to the Nash equilibrium.

Throughout this dissertation our main focus was to prove Theorem 1 and, show that the dynamics of 2×2 fictitious play is of finite type. In order to do that and to better understand this type of games we concentrated on the construction of an exhaustive list with all the possible combinatorially distinct types.

This study was done throughout chapter 4. Before that we clarified, from a dynamical point of view, the relation between the discrete and continuous fictitious play. These introductory and more theoretical chapters were the essential basis to achieve the main goal of this dissertation.

Based on this study we were able to present a family of examples of 2×2 fictitious play where we synthesized all the possible scenarios.

Regarding further research, a possible direction could be the study of games whose players have more than two pure strategies (see Sparrow (2008) and Shapley (1964)). For example the case where each one of the two players has three pure strategies. We know that, for these cases the convergence to the Nash equilibrium is not guaranteed. One of these cases is the example known as Shapley polygon where the fictitious play converges to a periodic orbit. Furthermore, it may not be possible to classify the dynamics with more than two pure strategies. In fact, in some games one may be in a presence of chaotic play. This opens a new area of study where many challenging problems are still waiting to be solved.

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Appendix A

Figures

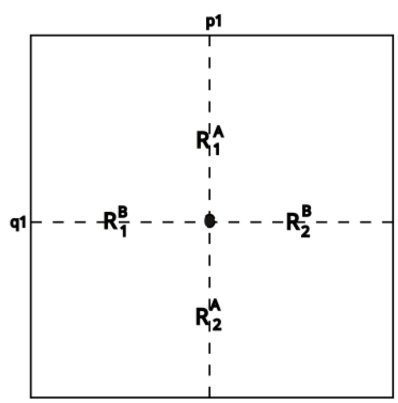

Figure A.1: Preference Regions

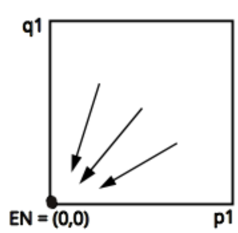

Figure A.2: Phase Portrait : Sink

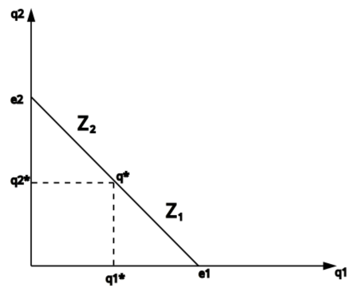

Figure A.3: Indifference Set

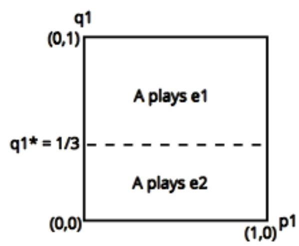

Figure A.4: Phase Portrait explaining the strategies of player A

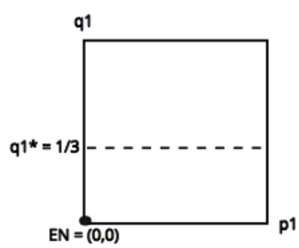

Figure A.5: Phase Portrait explaining the Nash Equilibrium

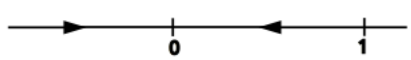

Figure A.6: convergence for q_1

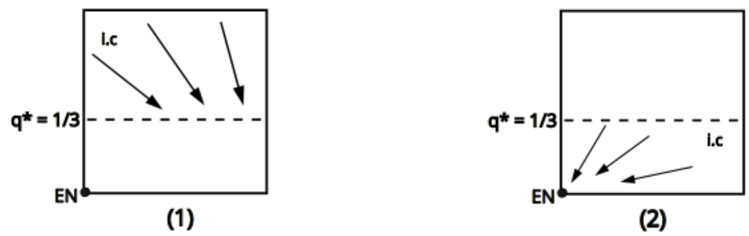

Figure A.7: (1)- Trajectory convergence when initial condition has $q_1 \geq \frac{1}{3}$. (2)- Trajectory convergence when initial condition has $q_1 \leq \frac{1}{3}$.

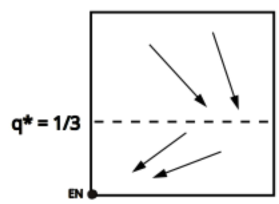

Figure A.8: Trajectory convergence over time

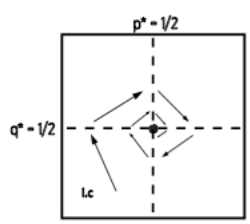

Figure A.9: Trajectory over time when initial condition is in first quadrant

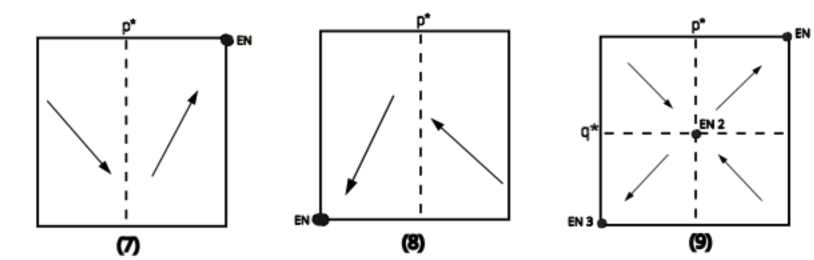

Figure A.10: Combinatorially cases for scenario (A)

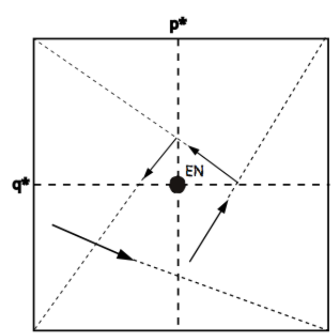

Figure A.11: case 9 in scenario (B)

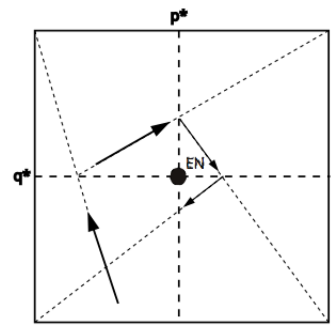

Figure A.12: case 9 in scenario (C)

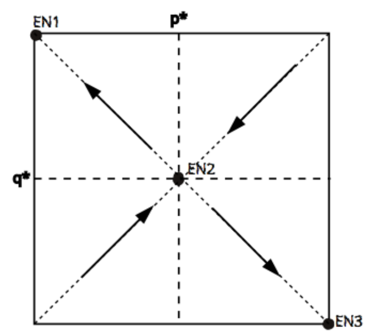

Figure A.13: case 9 in scenario (D)